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Inhomogeneous Mean Field Models

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The infinite set of coupled mean field equations for a classical inhomogeneous Ising ferromagnet is studied with respect to existence and uniqueness of its solutions.

KEY WORDS: Mean fields; inhomogeneous Ising model; convexity.

1. INTRODUCTION

We consider classical non-translation-invariant ferromagnetic lattice models. For some really interesting models one is able to derive partial results, but obtaining the full solution remains in general a pious hope. Yet, some idea and insight into the behavior of the model (e.g., bounds on the critical temperature[(1–3)]) can be obtained from the so-called mean field approximation.

In the case of translation-invariant systems this procedure is well known and leads to the solution of a set of equations for a finite number of spin variables (e.g., the components of the spin). These equations represent the Euler equations extremizing the free energy over a subset of states, namely, the product measures. These are the well-known self-consistency equations.

Here we are interested in the non-translation-invariant situation. Physically these models describe local perturbations of translation-invariant systems or inhomogeneous-temperature systems. They might show a phase transition even in one dimension with short-range interactions.[(4)] The particular mean field form of these models is defined in Section 2. These

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systems yield an infinite set of coupled equations [see Eqs. (2) below]. In terms of the interaction parameters we prove the existence and uniqueness of the constant sign solutions of these equations (see Theorem 1) and obtain the critical temperature explicitly in terms of the coupling constants. This result can be compared with the lower bound of the exact critical temperature of the original models.[5] Furthermore, alternating sign solutions are not excluded and are discussed in Section 4.

2. MEAN FIELD MODELS

Consider the lattice $\mathbb{Z}$, where a single spin is situated at each lattice site. The configuration space of the system is $X = \{-1, 1\}^{\mathbb{Z}}$ and let the spin observables σ_k be the functions mapping $x = (x_k) \in X \to x_k$.

Let μ be any product measure on X; define the local model Hamiltonian for any finite interval $\Lambda \subset \mathbb{Z}$

$$H_\Lambda^\mu = -\sum_{i\in\Lambda}\left[J_i\mu(\sigma_{i+1}) + J_{i-1}\mu(\sigma_{i-1})\right]\sigma_i \tag{1}$$

where J_i represents the interaction energy between the sites i and $i+1$. We restrict ourself to the case that $J_i > 0$ for all $i \in \mathbb{Z}$.

The Gibbs state determined by the Hamiltonian (1) defines a product measure on X leading in the usual way to the self-consistent equations for $\mu(\sigma_k)$ $(k \in \mathbb{Z})$:

$$\mu(\sigma_k) = \operatorname{th}\beta\left(J_k\mu(\sigma_{k+1}) + J_{k-1}\mu(\sigma_{k-1})\right), \qquad k \in \mathbb{Z} \tag{2}$$

We are interested in the solutions of Eqs. (2). One checks readily the trivial facts that $\mu(\sigma_k) = 0$ for all $k \in \mathbb{Z}$ is a solution and that, if $\mu(\sigma_k) \neq 0$ is a solution, then also $-\mu(\sigma_k)$ is a solution.

3. POSITIVE SOLUTIONS

We restrict ourself first to solutions of (2) such that for all $k \in \mathbb{Z}: \mu(\sigma_k) \geqslant 0$. The local interaction energies J_i $(i \in \mathbb{Z})$ define a linear operator J on the space $\mathbb{C}^{\mathbb{Z}}$ as follows:

$$(Ja)_k = \beta(J_{k-1}a_{k-1} + J_k a_{k+1}) \tag{3}$$

for all $a \in \mathbb{C}^{\mathbb{Z}}$. In this notation, the equations (2) become

$$a_k = \operatorname{th}(Ja)_k, \qquad k \in \mathbb{Z}, \qquad a \geqslant 0 \tag{4}$$

where $a \geqslant 0$ stands for $a_k \geqslant 0$ for all $k \in \mathbb{Z}$. If $Jl^2(\mathbb{Z}) \subseteq l^2(\mathbb{Z})$, denote by $\|J\|$ the norm of J restricted to $l^2(\mathbb{Z})$. Now we formulate our main result:

Theorem 1. The equations (4) admit at most two solutions. Furthermore

(i) $\|J\| \leqslant 1$ if and only if (4) admits the unique solution $a = 0$.
(ii) If $\|J\| > 1$ or if J is an unbounded operator on $l^2(\mathbb{Z})$ [i.e., $Jl^2(\mathbb{Z}) \not\subset l^2(\mathbb{Z})$], then (4) admits exactly two solutions ($a^1 = 0$ and $a^2 > 0$).

We prove this theorem in the following steps.

Lemma 2. If a and b are solutions of (4) then there exists a third solution c such that $a \leqslant c$ and $b \leqslant c$.

Proof. Let

$$V_{a,b} = \{d \in \mathbb{R}^{\mathbb{Z}} \mid d \geqslant a, d \geqslant b, 1 \geqslant d\}$$

Consider the map F from $\mathbb{R}^{\mathbb{Z}}$ into itself defined by

$$F(d)_k = \operatorname{th}(Jd)_k, \qquad d \in \mathbb{R}^{\mathbb{Z}}$$

then F is continuous for the product topology. Remark also that the set $V_{a,b}$ is compact and convex and as for $d \in V_{a,b}$:

$$F(d)_k = \operatorname{th}(Jd)_k \geqslant \operatorname{th}(Ja)_k = a_k$$

and

$$F(d)_k \geqslant b_k$$

$V_{a,b}$ is invariant for F

By a fixed point theorem (Ref. 6, Theorem V.19) we get the desired solution $c \in V_{a,b}$. ∎

Lemma 3. If a and b are solutions of (4) such that $0 < a \leqslant b$ then $a = b$.

Proof. Suppose $a \neq b$. As $0 < a_k \leqslant b_k$ ($k \in \mathbb{Z}$) by convexity

$$\operatorname{th}(Ja)_k = a_k = \frac{a_k}{b_k} b_k = \frac{a_k}{b_k} \operatorname{th}(Jb)_k \leqslant \operatorname{th} \frac{a_k}{b_k} (Jb)_k$$

Hence

$$(Ja)_k \leqslant \frac{a_k}{b_k} (Jb)_k$$

or

$$I_k \equiv J_k(a_{k+1}b_k - a_k b_{k+1}) + J_{k-1}(a_{k-1}b_k - a_k b_{k-1}) \leqslant 0 \tag{5}$$

Remark that the function

$$k \mapsto \frac{a_k}{b_k} \tag{6}$$

is monotonic or that there exists at most one index $k_0 \in \mathbb{Z}$ such that the function is monotonically increasing for $k \leqslant k_0$ and monotonically decreasing for $k \geqslant k_0$. Indeed, suppose that there exists $k_1 \in \mathbb{Z}$ such that

$$\frac{a_{k_1-1}}{b_{k_1-1}} \geqslant \frac{a_{k_1}}{b_{k_1}} \leqslant \frac{a_{k_1+1}}{b_{k_1+1}}$$

then from (5)

$$\frac{a_{k_1-1}}{b_{k_1-1}} = \frac{a_{k_1}}{b_{k_1}} = \frac{a_{k_1+1}}{b_{k_1+1}}$$

Let $\lambda = a_{k_1}/b_{k_1}$ then

$$a_{k_1} = \mathrm{th}(Ja)_{k_1} \qquad \text{and} \quad \lambda a_{k_1} = \mathrm{th}(J\lambda a)_{k_1}$$

Hence from the strict convexity of th: $\lambda = 0$ or $\lambda = 1$. The case $\lambda = 0$ is excluded because then $a = 0$ from (4). Furthermore if $\lambda = 1$, the equations (4) yield $a = b$. Suppose first that the function (6) is monotonically decreasing for $k \geqslant k_0$, i.e.,

$$a_{k-1}b_k - a_k b_{k-1} \geqslant 0, \qquad k \geqslant k_0 + 1 \tag{7}$$

From (5) for $k \geqslant k_0 + 1$

$$\begin{aligned}\sum_{l=0}^{n} I_{k+l} = {} & J_{k-1}(a_{k-1}b_k - a_k b_{k-1}) \\ & + J_{k+n}(a_{k+1+n}b_{k+n} - a_{k+n}b_{k+1+n}) \leqslant 0\end{aligned} \tag{8}$$

Suppose there exists a sequence $\{n_j\}_j$ with n_j tending to infinity such that

$$\lim_j J_{k+n_j}(a_{k+1+n_j}b_{k+n_j} - a_{k+n_j}b_{k+1+n_j}) = 0 \tag{9}$$

then from (8) and (7)

$$a_{k-1}b_k - a_k b_{k-1} = 0$$

and again from (4): $a = b$.

It remains to prove the existence of such a sequence. As the function (6) is decreasing the following limit exists:

$$\lim_{n\to\infty} \frac{a_{k+n}}{b_{k+n}}$$

Hence

$$\lim_{n\to\infty} (a_{k+1+n}b_{k+n} - a_{k+n}b_{k+1+n}) = 0$$

Therefore if there exists a sequence $\{n_j\}_j$ such that $\sup_{n_j>0} J_{n_j} < \infty$, (9) is satisfied. On the other hand suppose $\lim_{n\to\infty} J_n = \infty$. From (4) and the convexity of th

$$\operatorname{th} J_k \leqslant \frac{a_{k+1}}{a_k} \leqslant \frac{1}{\operatorname{th} J_k}$$

$$\operatorname{th} J_k \leqslant \frac{b_{k+1}}{b_k} \leqslant \frac{1}{\operatorname{th} J_k}$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty} J_{k+n}|a_{k+1+n}b_{k+n} - a_{k+n}b_{k+1+n}| \\
&\quad = \lim_{n\to\infty} J_{k+n}a_{k+n}b_{k+n}\left|\frac{a_{k+1+n}}{a_{k+n}} - \frac{b_{k+1+n}}{b_{k+n}}\right| \\
&\quad \leqslant \lim_{n\to\infty} J_{k+n}\left(\frac{1}{\operatorname{th} J_{k+n}} - \operatorname{th} J_{k+n}\right) = 0
\end{aligned}$$

Finally, the case that the function (6) is monotonically increasing is treated analogously. The result is now obtained by summing up (5) but for $n < 0$. ∎

In order to get the proof of the first statement of Theorem 1, it is now sufficient to prove the following:

Lemma 4. If a is a solution of (4) such that $a \neq 0$, then $a > 0$.

Proof. Suppose that $a_{k_0} = 0$; then from (4), $a_{k_0-1} = a_{k_0+1} = 0$ and by recursion $a = 0$. Hence all $a_k > 0$. ∎

Now we proceed to the proof of (i) and (ii) of Theorem 1. For $n, m \in \mathbb{Z}$ and $n < m$, denote by $P^{n,m}$ the projection of $\mathbb{C}^{\mathbb{Z}}$ onto $\mathbb{C}^{\{n,n+1,\dots,m\}}$ and by

$$J^{n,m} = P^{n,m} J P^{n,m}$$

Lemma 5. If $\|J\| > 1$ or $Jl^2(\mathbb{Z}) \not\subset l^2(\mathbb{Z})$ then there exist n and m in $\mathbb{Z}, n < m$ such that $\|J^{n,m}\| > 1$.

Proof. If $\|J^{n,m}\| \leqslant 1$ for all $n, m \in \mathbb{Z}$, as for all $x \in P^{n,m}\mathbb{C}^{\mathbb{Z}}$

$$Jx = J^{n-1,m+1}x$$

it follows that $\|J\| \leqslant 1$. ∎

Lemma 6. If $\|J\| > 1$ or $Jl^2(\mathbb{Z}) \not\subset l^2(\mathbb{Z})$ then there exists a solution $a \neq 0$ of (4).

Proof. From Lemma 5 there exist n and m in $\mathbb{Z}$, $n < m$ such that $\|J^{n,m}\| > 1$. From Perron's theorem there exists a vector $z \in P^{n,m}\mathbb{C}^{\mathbb{Z}}$ such that $\|z\| = 1$, $z_k \geqslant 0$ for all $k \in [n,m]$ and

$$J^{n,m}z = \|J^{n,m}\|z$$

Let $k \in [n,m]$, as $\|J^{n,m}\| > 1$ the equation

$$\mu z_k = \text{th } \mu(J^{n,m}z)_k = \text{th } \mu\|J^{n,m}\|z_k$$

has at least one solution $\mu = \lambda_k > 0$. Let $\lambda = \min_{k\in[n,m]}\lambda_k > 0$; then as $Jz \geqslant J^{n,m}z$:

$$\lambda z_k \leqslant \text{th}\lambda(Jz)_k, \qquad k \in \mathbb{Z}$$

Now using the fixed point theorem on the function F as in the proof of Lemma 2 to the region

$$V_{\lambda z} = \left\{x \in \mathbb{R}^{\mathbb{Z}} \mid \lambda z_k \leqslant x_k \leqslant 1\right\}$$

one gets a nonzero solution of (4). ■

Finally the proof of theorem 1 is completed by the following lemma.

Lemma 7. Suppose that a is a nonzero solution of (4), then $\|J\| > 1$ or $Jl^2(\mathbb{Z}) \not\subset l^2(\mathbb{Z})$

Proof. Suppose $\|J\| \leqslant 1$ and a is a solution of (4) such that $a \neq 0$. From (4) we have

$$a_k = \text{th}(Ja)_k \leqslant (Ja)_k \tag{10}$$

If $a \in l^2(\mathbb{Z})$ then

$$((J-1)a, a) \geqslant 0$$

As $J = J^*$ and $\|J\| \leqslant 1$:

$$\sum_k \left[(Ja)_k - a_k\right]a_k = 0$$

By Lemma 4: $a_k > 0$, therefore

$$(Ja)_k = a_k = \text{th}(Ja)_k$$

hence $a_k = 0$, which is a contradiction. Suppose now that $a \notin l^2(\mathbb{Z})$. From (10) again

$$\begin{aligned} 0 &\leqslant \left(P^{i,j}(J-1)a, P^{i,j}a\right) \\ &= \left((J-1)P^{i,j}a, P^{i,j}a\right) + J_{i-1}a_{i-1}a_i + J_j a_{j+1}a_j \\ &\leqslant J_{i-1}a_{i-1}a_i + J_j a_{j+1}a_j \end{aligned} \tag{11}$$

Suppose that there exist sequences $\{j_k\}$ tending to plus infinity and $\{i_k\}$

tending to minus infinity such that

$$\lim_k \left(J_{i_k-1}a_{i_k-1}a_{i_k} + J_{j_k}a_{j_k+1}a_{j_k}\right) = 0$$

then from (11) again $a_k = 0$ by the argument above.

Suppose now that there exists $\epsilon > 0$ such that for all i, j

$$J_{i-1}a_{i-1}a_i + J_j a_{j+1}a_j \geqslant 2\epsilon$$

Then there exists a sequence $\{i_k\}$ or $\{j_k\}$ tending to minus or plus infinity such that

$$(Ja)_{i_k} \geqslant J_{i_k-1}a_{i_k-1}a_{i_k} \geqslant \epsilon$$

or the corresponding inequalities for the j_k's. Observing that the function $y - \mathrm{th}\, y (y \in \mathbb{R})$ is increasing one gets

$$\begin{aligned}(Ja)_{i_k} - a_{i_k} &= (Ja)_{i_k} - \mathrm{th}(Ja)_{i_k}\\ &\geqslant \epsilon - \mathrm{th}\,\epsilon\end{aligned}$$

Also $a_{i_k} = \mathrm{th}(Ja)_{i_k} \geqslant \mathrm{th}\,\epsilon$.
Therefore using (10)

$$\left(P^{i_k,j}(J-1)a, P^{i_k,j}a\right) \geqslant k(\epsilon - \mathrm{th}\,\epsilon)\mathrm{th}\,\epsilon$$

which contradicts (11) as $k \to \infty$. ■

4. ALTERNATING SIGN SOLUTIONS

First of all we warn the reader here that we have no claims of characterizing in full detail the complete set of solutions of Eq. (2). We provide some examples of systems with alternating sign solutions. They show that in general the set of solutions can be very large, containing solutions of quite different types.

The only general result we can state is the following:

Proposition 8. If a is a nonzero positive solution of Eq. (2) then any other solution b satisfies

$$|b_k| \leqslant a_k \qquad \text{for all} \quad k \in \mathbb{Z}$$

Proof. By repeating literally the proof of Lemma 2 we get the existence of a positive solution c such that $a \leqslant c$ and $b \leqslant c$. By Theorem 1, $a = c$. Hence $b \leqslant a$. Observe now that $-a$ is the unique nonzero negative solution. Consider now the set $V_{-a,b} = \{d \in \mathbb{R}^{\mathbb{Z}} \mid d \leqslant -a, d \leqslant b, -1 \leqslant d\}$. By the same argument as above: $b \geqslant -a$. ■

This property implies that there are no solutions with alternating sign if zero is the unique nonnegative solution. Hence nonzero alternating sign solutions occur only at lower temperatures if they occur at all.

Finally we give some examples of solutions of nonconstant sign. Periodic solutions are easily obtained for periodic systems, e.g., take the J_k such that

$$J_k = J_{k+6} \quad \text{for all} \quad k \in \mathbb{Z}$$
$$J_0 = J_5, \qquad J_2 = J_3, \qquad J_1 = J_4 > 1$$

Then a solution of Eq. (2) is given by $a_k = a_{k+6}$ for all $k \in \mathbb{Z}$, $a_0 = a_3 = 0$ and $a_1 = a_2 = -a_4 = -a_5 = \alpha$ where α is the nonzero solution of $\alpha = \operatorname{th} J_1 \operatorname{th} J_1 \alpha$.

Note that these examples extend to higher period solutions. Furthermore the translation-invariant case is included in these systems.

For non-translation-invariant systems there are also systems with alternating sign solutions but not periodic, e.g., take the system

$$J_j = J > 1 \qquad \text{for} \quad k = 0, 1, 2, \ldots \quad \text{and} \quad k = -3, -4, \ldots$$

and J_{-1} and $J_{-2} > 1$ to be determined later on. Take the nontrivial solution of $\alpha = \operatorname{th} J \operatorname{th} J\alpha$. Consider again the fixed point solution b in $V_a = \{d \in \mathbb{R}^{N_0} | d \geqslant a\}$, where a is now $(\alpha, \alpha, 0, 0, 0, \ldots)$, of the mapping

$$F(b)_1 = \operatorname{th} J b_2$$
$$F(b)_k = \operatorname{th} J(b_{k+1} + b_{k-1}), \qquad k \geqslant 2$$

Clearly $b_1 > 0$. Take the negative solution b_{-1} of $b_{-1} = \operatorname{th} J_{-2} \operatorname{th} J_{-2} b_{-1}$ and choose $J_{-1} = -Jb_1/b_{-1}$; then

$$(\ldots, b_2, b_1, 0, b_{-1}, b_{-1}, 0, b_1, b_2, \ldots)$$

is a solution of Eq. (2).

Such solutions do remember Dobrushin's result[7] for the Ising model in three or more dimensions where equilibrium states are constructed with magnetization changing from m to $-m$ and further work on non-translation-invariant states.[8,9] In this context our result may be relevant as a one-dimensional molecular field model for an interface.

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